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A Study on linear dynamics of axially moving string

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ABSTRACT

Axially moving materials can represent many engineering devices such as power transmission belts, elevator cables, plastic films, magnetic tapes, textile fibers and band saws. But practically the transverse vibration and the noise associated with it have limited the applications despite of all its advantages. Therefore, understanding transverse vibrations of axially moving strings is important for the design of many devices. Hence the paper is a review of linear free, forced and parametric vibration associated with an axially moving string. The process involves derivation of equation of motion, determination of natural frequency, understanding the nature of frequency and mode shape, generating response for linear free and forced vibration and determining instability regions due to parametric vibration in an axially moving string.

KEY WORDS: Axially moving string, parametric vibration.

1. INTRODUCTION

The investigations on transverse vibrations and control of axially moving strings have theoretical significance, because an axially moving string is a simplest representative of distributed gyroscopic system. Li Qun Chen in his review paper talks about linear, non-linear and parametric vibration of axially moving string in detail. The dynamical equation was derived from Newton's second law of motion while dealing with a moving threadline and named the equation as "threadline equation". They applied the theory of characteristics to explore the nature of wave propagation in the string under boundary excitation and predicted erratic string behavior near a critical speed. Furthermore the natural frequency of each mode decreases with the transport speed. A convective acceleration component in the equations of motion results in complex, speed-dependent modes. The study was further carried forward on band saws and a theoretical analysis of band saw small vibrations consisting of the exact solution for the transverse vibration natural frequencies as well as bounding approximate solutions were found. He considered the band flexural rigidity and the band tension--velocity dependence for computing the results so that the results can be applied in general to all band types of band saws.

Despite the apparent simplicity of the traveling string model, the response of either model to general excitation and initial conditions cannot be analytically predicted using previous methods for axially moving material. The orthogonality of the eigen functions cannot be derived. Accordingly, the generalized coordinates in an eigen function expansion remain coupled, and the classical modal analysis applied to the non-translating string and beam models, are not applicable to axially moving continua. The equations of motion for gyroscopic systems resemble those for viscously damped natural systems, with the exception that in the case of gyroscopic systems the matrix of the coefficients of the velocity terms is skew symmetric as opposed to damped systems for which it is symmetric. Whereas in certain special cases the classical modal analysis can diagonalize a viscously damped natural system, but under no circumstances can it diagonalize a gyroscopic system. Hence a new Modal Analysis technique for discrete gyroscopic system was formulated. In his method the system of equation was represented as two first order equations that satisfies the orthogonality relationships. The response of the system was then represented by applying expansion theorem. It is Wickert and Mote who first treated transverse vibration of a moving string by use of the modal analysis. They produced exact closed form expression of the response of a moving string to arbitrary excitation and initial conditions. They further demonstrated the complex nature of mode shape (eigen function). Another possible effect that occurs with axially moving string is a variation in tension or speed while in motion which leads to a special form of vibration called parametric vibration.

Dynamics of the axially moving string is reviewed at first. The first section describes the characteristics of an axially moving string when compared to a fixed string and also generalizes the modal analysis procedure for an axially moving string put forward by L.Meirowitch. Then the study is carried over to parametric vibration. Parametric vibration can happen in an axially moving string due to variation in tension alone, variation of velocity with constant tension and variation in velocity producing a variation in tension. This paper deals with parametric vibration due to variation of velocity which produces a variation in tension.

Free and forced vibration of an axially moving string: In the case of an axially moving string, in addition to the wave velocity one more velocity term i.e. axial velocity 'v' acts on the string. Fig1 is a schematic diagram of such a system. A segment of the string experiences a motion in both the x and y directions as it vibrate about the equilibrium

position. This is easily seen in fig 2 where the element of string labelled 'ds' moves in the x and y direction in a time interval 'dt'.

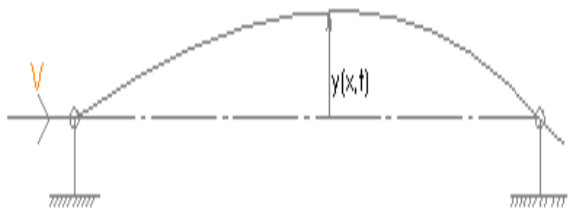

Figure.1. Axially moving string

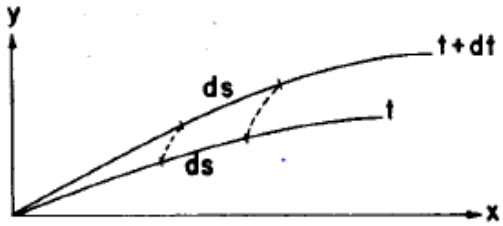

Figure.2. Transposed motion of an axially moving string

Deriving the Equation of Motion: Consider the motion of a threadline moving with a velocity 'v' between two eyelets spaced a distance L apart. The displacement of the string at any particular point is given by $y(x,t)$. On applying Newton's second law of motion we get the equation of motion as

$$\frac{\partial^2 y}{\partial t^2} + 2v\frac{\partial^2 y}{\partial x \partial t} + (v^2 - \frac{Tg_0}{\rho})\frac{\partial^2 y}{\partial x^2} = 0 \qquad (1)$$

where v = axial velocity, T = tension, ρ = mass per unit length. The governing equation of motion is found to be a second order partial hyperbolic equation. It is quite similar to that of a fixed string, only difference being the presence of two additional terms which denote coriolis($2v\frac{\partial^2 y}{\partial x \partial t}$) and centripetal acceleration ($v^2\frac{\partial^2 y}{\partial x^2}$). Because the coriolis acceleration is related to gyroscopic phenomena, most often associated with spinning bodies, the term linear in velocity is also referred to as gyroscopic term. Furthermore any system executing a transposed motion is called a gyroscopic system and here, the system is executing a transposed translation motion. For example,If one particle of a string is considered during motion, it demonstrates vertical as well as horizontal motion at the same time and hence the system is an example of a simple distributed gyroscopic system.

Frequency and Mode Shape: Considering the string to be supported at both the ends, the boundary condition can be taken as

$$y(0,t) = 0 \text{ and } y(l,t) = 0 \qquad (2)$$

Applying separation of variables

$$y(x,t) = W(x)e^{i\omega t} \qquad (3)$$

where $W(x)$ is the eigenfunction and ω circular frequency

to the equation of motion, it becomes

$$W''(x)e^{iwt} + 2i\omega v W'(x)e^{i\omega t} - \left(v^2 - \frac{Tg_0}{\rho}\right)\omega^2 W(x)e^{i\omega t} = 0 \qquad (4)$$

Substituting W(x) = Be^{ikx} we get the eigen value problem for a travelling string as

$$-(c^2 - v^2) - k^2 Be^{ikx} + 2i\omega vikBe^{ikx} - \omega^2 Be^{ikx} = 0 \qquad (5)$$

which upon solving gives

$$k = \frac{\omega}{c-v}, \frac{-\omega}{c+v} \qquad (6)$$

Hence the general solution can be taken as

$$W(x) = De^{\frac{-i\omega x}{c+v}} + Ee^{\frac{i\omega x}{c-v}} \qquad (7)$$

which upon solving gives the frequency as

$$\omega_n = \frac{n\pi(c^2 - v^2)}{2lc} \qquad (8)$$

This gives the natural frequency of a string undergoing axial motion. The variation of the first three frequencies with speed of travel is shown in fig 3. It is interesting to see that all the frequencies are zero when the translation speed equals the wave speed of the string. And from the governing equation it can be found that the string

loses its stiffness completely at this point. Hence, this speed is known as the critical speed of translation of the string. From (8) it is clear that factors affecting the natural frequency are mode number., axial velocity and wave velocity. Substituting the frequency into the (7) and solving for D and E gives the mode shape.

$$W(x) = D_n.e^{\frac{in\pi xv}{lc}}[\sin\frac{n\pi x}{l}] \qquad (9)$$

The equation gves the eigen function which is complex in nature. So an axially moving string is found to possess complex mode shape. It is mathematically explained in the coming section. The real and imaginary components of the mode shape are as follows

Real Part : $W_r(x)=\sin(n\pi x/l)\cos(n\pi vx/lc)$ (10)

Imaginary Part: $W_i(x)=\sin(n\pi vx/lc)\sin(n\pi x/l)$ (11)

Fig 4 shows the complex mode shape at different values of $\gamma = 0.5$ where ($\gamma = \frac{v}{c}$). After checking for different values of γ ,no particular pattern is found in mode shape.

Response for free and forced vibration of an axially moving string: The governing equation of motion gets reduced to the form as shown in (12) on non-dimensionalizing.

$$\frac{\partial^2 y}{\partial t^2} + 2\upsilon\frac{\partial^2 y}{\partial x \partial t} - (1-\gamma^2)\frac{\partial^2 y}{\partial x^2} = 0 \qquad (12)$$

This non-dimensionalized equation can be represented as

$$Mu_{tt} + G'u_t + Ku = 0 \qquad (13)$$

where $M = I'; G' = 2v\frac{\partial}{\partial x}; K = -(1-\gamma^2)\frac{\partial^2}{\partial x^2}$ (14)

The modal analysis technique applied for a fixed string is not applicable here due to the presence of a gyroscopic term. Even though it seems to be similar to the equation of a system with damped vibration which can be diagonalized , under no circumstances the coefficient of velocity terms for a gyroscopic term can be diagonalized. It was then, Meirowitch came up with a new modal analysis technique which helps to find a closed form solution to linear gyroscopic systems. The procedure is as follows

- Non-dimensionalise the equation of motion. It can be represented in a generalized form as

$$M\ddot{y}(t) + G'\dot{y}(t) + Ky(t) = X(t) \qquad (15)$$

where $X(t)$ denotes external excitation

Represent the equation of motion as 2n X 2n real non-singular matrices

$$IW_t + GW = 0 \qquad (16)$$

$$\left.\begin{array}{l} \text{where } I = \begin{pmatrix} M & 0 \\ 0 & K \end{pmatrix} \text{ and } G = \begin{pmatrix} G' & K \\ -K & 0 \end{pmatrix} \\ \\ \text{with state vectors } W(x,t) = \begin{pmatrix} u_t \\ u \end{pmatrix} \end{array}\right\} \qquad (17)$$

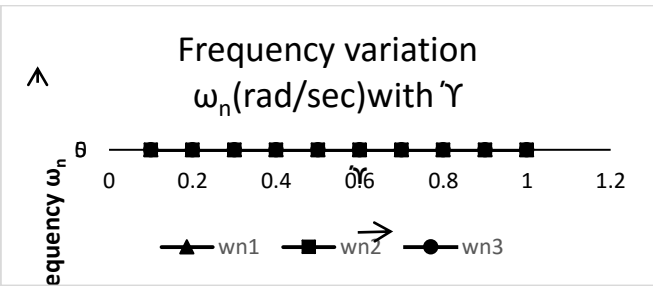

Figure.3.Variation of natural frequency with axial velocity

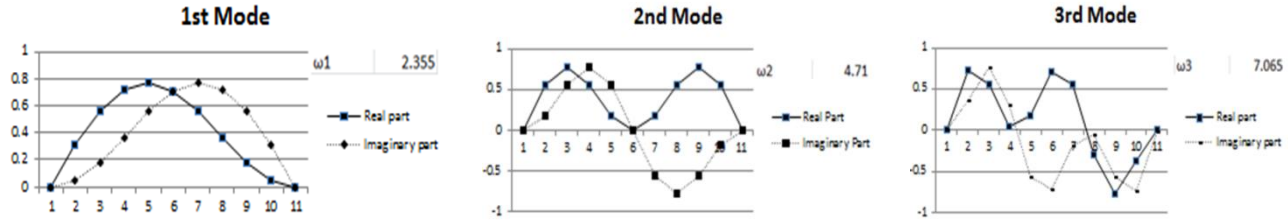

Figure 4.Mode shape at $\gamma = 0.5$

- Apply variable separable form $W_x(t) = e^{\lambda t}\phi(x)$, the equation get reduced to

$$I\lambda\phi(x) + G\phi(x) = 0 \qquad (18)$$

- Reduce the skew-symmetric matrix to symmetric one.

G being skew symmetric, the eigen values and eigen vectors are complex. Hence substitute the complex eigen value and vector to the equation (18) which helps to separate the real and imaginary part $\omega_r^2 Iy_r = Ky_r 1$ and $\omega_r^2 Iz_r = Kz_r$ where $K = G^T I^{-1} G$

(19)

This explains the presence of real and imaginary eigen functions. Using the relation shown in (16) and (19) the G matrix can also be diagonalized.

- Use the orthogonality relationship and expansion theorem to find the final generalized response.
- If there is a forcing function, substitute X (t) as that function and expand to get the final response.

Using this method the response for free vibration is obtained as

$$x(t) = \sum_{r=1}^{n} \int_0^t a_r d\tau + b_r$$

where

$$a_r = X(\tau)\left[\cos\omega_r(t-\tau)\left(y_r y_r^T + z_r z_r^T\right) + \sin\omega_r(t-\tau)\left(y_r z_r^T - z_r y_r^T\right)\right] \qquad (20)$$

$$b_r = Ix(0)\left[\cos\omega_r t\left(y_r y_r^T + z_r z_r^T\right) + \sin\omega_r t\left(y_r z_r^T - z_r y_r^T\right)\right]$$

For the case of harmonic excitation, let the excitation vector be of the form ($X(t) = X\cos(\omega t)$). Assuming the initial conditions to be zero, we get the response as

$$x(t) = \sum_{r=1}^{n} \frac{1}{\omega^2 - \omega_r^2}[(y_r y_r^T + z_r z_r^T)X(\omega\sin(\omega t) - \omega_r\sin(\omega_r t)) + (y_r z_r^T - z_r y_r^T)X\omega_r(\cos(\omega_r t) - \cos(\omega t))] \qquad (21)$$

Parametric Vibration of An Axially Moving String: Periodic variation of some of the parameters in the equation of motion results in parametric vibration. It is a special class of self-excited vibration. For example, a rotating rectangular shaft (or any other irregular shaft), a variable length pendulum are examples of parametric vibration due to variable elasticity. In most of the problems, the equation of motion reduces to Matheiu's equation for which exact solution is not known. However we are not so much interested in the solution itself, the objective is to know whether the solution is stable or unstable and plotting a Strutt diagram as described by Den Hartog (1934) helps to differentiate between stable and unstable regions

For an axially moving string, parametric vibration arises mainly due to two factors- tension variation keeping velocity a constant and velocity variation keeping tension a constant or velocity-dependent tension. The paper reviews parametric vibration due to varying velocity and velocity -dependent tension. Pakdemirli (1994) did stability analysis of an accelerating string using Flouquet theory.

Here a stability analysis is performed for 1 and 2-term Galerkin's approximation solutions using time-integration method. Different points are selected in the region under consideration and stability was checked. It was found that Galerkin's 1-term approximation leads to a Matheiu's equation, the solution of which is quite well known whereas 2-term approximation produces a gyroscopically coupled equation.

Equation of Motion: The equation of motion is derived using the Newton's second law of motion as was done in the section (2.1) and it was obtained as

$$\rho A(\ddot{y} + \dot{v}y' + 2v\dot{y}') + (\rho A v^2 - P)y'' = 0 \qquad (22)$$

where ρ is the density, A represents cross sectional area, v is the axial velocity, $\dot{v}$ being acceleration and P denotes tension.

Solution Method: Galerkin's method is used to solve the equation of motion. The trial function is taken as

$$y(x,t)=\sum_{i=1}^{n} q_i(t)\sin(\frac{i\pi x}{L}) \qquad (23)$$

where $\sin(\frac{i\pi x}{L})$ is the ith eigen function of a simply supported stationary string. It is done so as to reduce the complexity that would arise upon using eigenfunction of a moving string.

Galerkin's 1-Term Approximation: Taking 1-term approximation gives a trial function of the form

$$y(x,t)=q_1(t)\sin(\frac{\pi x}{L}) \qquad (24)$$

Substituting the trial function in the equation of motion produces a residual

$$R=[\rho A\ddot{q}_1\sin(\frac{\pi x}{l})+\frac{2\rho Av\pi\dot{q}_1}{L}\cos(\frac{\pi x}{l})+\frac{\rho A\dot{v}\pi i q_1}{L}\cos(\frac{\pi x}{l})+ (P-\rho Av^2)\frac{\pi^2 q_1}{L^2}\sin(\frac{\pi x}{L}) \qquad (25)$$

Taking the weighted residual as $w_j(x)=\sin(\frac{\pi x}{L})$ and applying Galerkin's Weighted Residual Method

$$\int_0^L Rw_j(x)dx=0 \qquad (26)$$

reduces the (22) to

$$\ddot{q}_1+(\frac{P}{\rho A}-v^2)\frac{\pi^2}{L^2}q_1=0 \qquad (27)$$

Considering a periodic variation of axial velocity and velocity dependent tension as

$$v(t)=v_0\sin(\omega_0 t) \quad \text{and} \quad P=P_0+\eta\rho Av^2 \qquad (28)$$

where v_0 is the axial velocity amplitude , ω_0 is the frequency of axial velocity variation, P_0 forms the initial tension and η represents pulley parameter. Upon substituting the (28) in (27) the solution transforms into

$$\frac{d^2q_1}{dt'^2}+(\delta+2\varepsilon\cos(2t')q_1=0 \qquad (29)$$

where $\delta=(\frac{2P_0}{\rho A}-kv_0)\frac{\pi^2}{2\omega_0{}^2L^2}$ and $\varepsilon=\frac{kv_0{}^2\pi^2}{4\omega_0{}^2L^2}$ (30)

The equation is called Mathieu's equation for which a standard solution is already available in the form of a Strutt diagram which differentiates between the stability and instability regions. The fig (5) shows a Strutt diagram plotted by using the Perturbation technique.

So based on the Strutt diagram stability at different random points are checked using the time-integration method after applying values to the constant parameters in the (30) as P_0 = 76.22N, ρ = 7754kg/m3, A =0.5202*10^{-5} m^2, k=0.22 and L=0.3681m. The results are found to be in agreement with the Strutt diagram. Fig(6) and (7) demonstrates stable points whereas fig(8) denotes an unstable point.

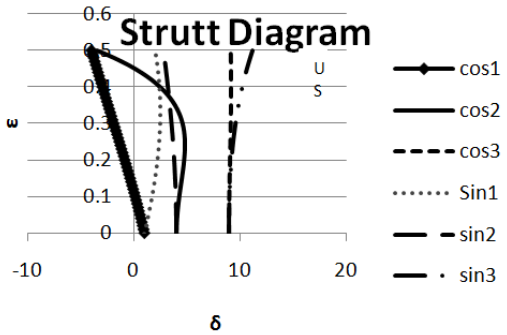

Figure.5.Strutt Diagram

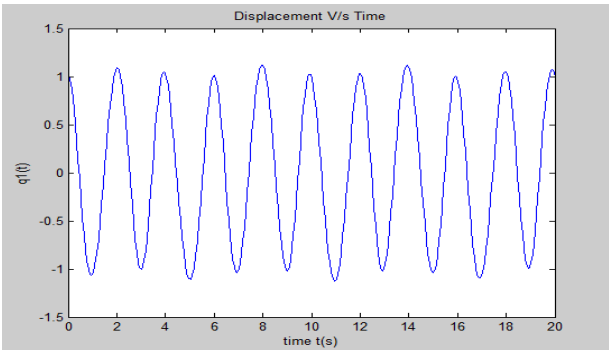

Figure.6.Displacement vs time for $\delta = 10$ and $\varepsilon = 1$

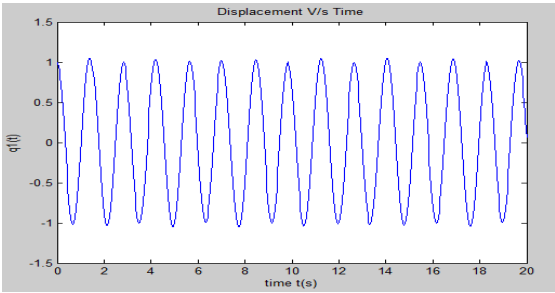

Figure.7.Displacement vs time for $\delta = 20$ and $\varepsilon = 1$

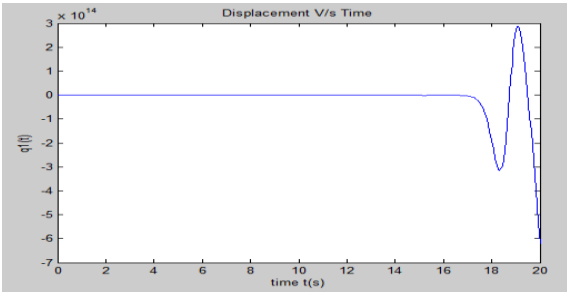

Figure.8.Displacement vs time for $\delta = 1$ and $\varepsilon = 10$

Galerkin's 2-Term Approximation: The procedure followed for the 2-term approximation is same as that of 1-term approximation. After applying the Galerkin's Weighted Residual method, the equation (22) becomes a gyroscopically coupled equation as shown in equation (31)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}1 \\ \ddot{q}2 \end{bmatrix} + \begin{bmatrix} 0 & -\dfrac{16v_0 \sin\omega_0 t}{3L} \\ \dfrac{16v_0 \sin\omega_0 t}{3L} & 0 \end{bmatrix} \begin{bmatrix} \dot{q}1 \\ \dot{q}2 \end{bmatrix} + \begin{bmatrix} (\dfrac{2P_0}{\rho A} - kv_0^{\,2} + kv_0^2 \cos 2\omega_0 t)\dfrac{\pi^2}{2L^2} & -\dfrac{8v_0\omega_0 \cos(\omega_0 t)}{3L} \\ \dfrac{8v_0\omega_0 \cos(\omega_0 t)}{3L} & (\dfrac{2P_0}{\rho A} - kv_0^{\,2} + kv_0^2 \cos 2\omega_0 t)\dfrac{2\pi^2}{L^2} \end{bmatrix} \begin{bmatrix} q1 \\ q2 \end{bmatrix} = 0 \qquad (31)$$

After applying corresponding values for all the constants, this equation can be solved using Runge-Kutta method after reducing the above equations to a set of four first order equations. The stability at individual points, which may be a set of v_0 and ω_0 or δ and ε can be found using this time-history analysis. The observation at some of the points are shown in Figure9 and Figure 10.

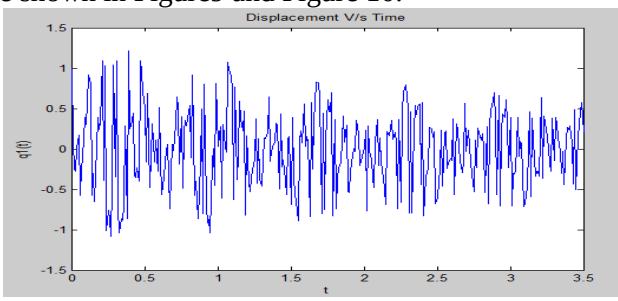

Figure.9.Displacement vs time for δ = 41 and ε =18

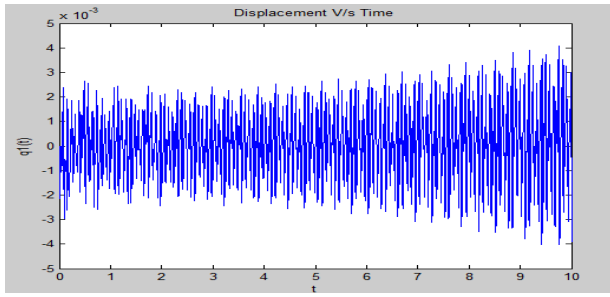

Figure.10.Displacement vs time for δ = 37 and ε =9

2. CONCLUSION

It was observed that the behavior of the string completely changes when it is set to motion. This peculiar is effectively reviewed in this paper. An axially moving string is a simplest representative of a distributed gyroscopic system. Any system executing a transposed motion is a gyroscopic system and the string during its motion is executing a transposed translational motion. It is the coriolis acceleration component experienced by axially moving materials which imparts a skew-symmetric or gyroscopic term to their governing equations. The presence of this gyroscopic term makes the system complex that is, it results in complex eigen values and complex eigen functions.

It is also found that the frequency of the string is affected by the axial velocity of the string. It appears that as the magnitude of axial velocity approaches that of wave velocity, the frequency of oscillation gets smaller, that is, the oscillations become slower and when axial velocity equals the wave velocity, the frequency of each mode dies out. One more interesting phenomenon at this condition is that the string loses its stiffness at this point. It was also found that the Classical modal analysis method fails with gyroscopic system as it is not possible for the eigenfunctions to normalize mass, stiffness and gyroscopic functions.

In the case of parametric vibration time-integration method is effectively implemented to find the stable and unstable points. It was found from the Galerkin's 1-term approximation that the possibility of instability is more with increasing speed, whereas the system is more stable at higher tension. As the parameter δ increases with lower ε, chances for stability increases. But when the parameter ε increases with lowering δ, the chances of getting unstable point is found to increase. This happens because the parameter v_0 appears in both and δ and ε, so increasing v_0 increases ε and decreases δ .Therefore there is a possibility of instability at higher speeds. But P_0 influences δ only. So increasing P_0 increases δ and hence increases stability. 2-term approximation gives better convergence.

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